TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 218, 1976

CERTAIN CONTINUA IN S^n OF THE SAME SHAPE HAVE HOMEOMORPHIC COMPLEMENTS(1)

BY

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ABSTRACT. As a consequence of Theorem 1 of this paper, we see that if X and Y are globally 1-alg continua in S^n $(n \geq 5)$ having the shape of the real projective space P^k $(k \neq 2, 2k + 2 \leq n)$, then $S^n - X \approx S^n - Y$. (For $P^1 = S^1$, this establishes the last case of such a result for spheres.) We also show that if X and Y are globally 1-alg continua in S^n, $n \geq 6$, which have the shape of a codimension ≥ 3, closed, $0 < (2m - n + 1)$-connected, PL-manifold M^m, then $S^n - X \approx S^n - Y$.

1. **Introduction.** The problem of classifying the shape of compacta in S^n (or E^n) in terms of their complements in S^n (or E^n) has been studied by a number of people.

In [3], Chapman proved that two Z-sets in the Hilbert cube have the same shape if and only if their complements are homeomorphic. Working with Z_k-sets, Geoghegan and Summerhill [10] improved the finite dimensional theorem of Chapman [4] by reducing the condition $n \geq 3k + 3$ to $n \geq 2k + 2$.

Rushing, in [16], proved that for a continuum X in S^n $(n \geq 5)$, $\mathrm{Sh}(X) = \mathrm{Sh}(S^k)$ (S^k is the standard k-sphere in S^n) is equivalent to $S^n - X \approx S^n - S^k$, if X is globally 1-alg in S^n and $k \neq 1$ ($S^n - X$ must have homotopy type of S^1 if $k = n - 2$). He also gave an example to show that $S^n - X \approx S^n - S^1$ is not sufficient to imply $\mathrm{Sh}(X) = \mathrm{Sh}(S^1)$.

In [5], Coram, Daverman and Duvall proved that if $\dim X \leq n - 3$, $X \subset E^n$ satisfies small loop condition (SLC) and X has the shape of a finite complex K in the trivial range, then X has a neighborhood N in E^n such that $N - X \approx \partial N \times [0, 1)$, where N is also a regular neighborhood of a copy of K in E^n $(n \geq 5)$.

Recently, Coram and Duvall have proved [6] the equivalence of $S^n - X \approx S^n - Y$ and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, where X, Y are sphere-like continua in S^n $(n \geq 5)$ (see definition in [15]) satisfying SLC and $\max\{\dim X, \dim Y\} \leq n - 4$.

Received by the editors February 17, 1975.

AMS (MOS) subject classifications (1970). Primary 57A15, 57A35; Secondary 57C20, 57C30, 57C40.

Key words and phrases. Stable end, H-cobordism, regular neighborhood, shape, globally 1-alg.

(1) This research will constitute a part of the author's doctoral dissertation under the direction of Professor T. B. Rushing at the University of Utah.

However, notice that the class of sphere-like continua is much smaller than the class of continua having the shape of sphere-like continua.

In this note, we consider globally 1-alg continua in S^n having either the shape of finite complexes in the trivial range or the shape of closed, simply connected PL-manifolds of codimension $\geqslant 3$. As a result, we solve the S^1-case of Rushing [16]. Thus, we generalize the main result of Daverman [8].

I am very grateful to Professor T. B. Rushing for many helpful questions and discussions concerning this paper. I also thank Dr. R. Stern for his help.

2. **Notation and definitions.** Throughout this note, we use the following notations:

$\approx$ Homeomorphic or isomorphic

$\simeq$ Homotopy equivalence or homotopic

$\sim$ Homologous

∂V, Int V Boundary, interior of a manifold V

i or $A \hookrightarrow B$ Inclusion map

$f_*, f_\#$ Induced maps on homotopy, homology groups

H_* Singular homology, Z coefficients

$\check{H}^*$ Čech cohomology, Z coefficients

For basic shape theory results, we refer to [1] and [15]. For convenience, in this paper we use both shape theories [1] and [15] as is justified in [22].

A continuum X in S^n is said to be *globally* 1-alg in S^n if for every neighborhood U of X in S^n, there is a neighborhood V of $X (V \subset U)$ such that if $f: S^1 \longrightarrow V - X$, $f \sim 0$ in $V - X$, then $f \simeq 0$ in $U - X$.

An inverse sequence of groups

$$G_1 \overset{i_1}{\longleftarrow} G_2 \overset{i_2}{\longleftarrow} \cdots$$

is said to be *constant* if we have

$$\operatorname{Im} i_1 \underset{\approx}{\overset{i_1}{\longleftarrow}} \operatorname{Im} i_2 \underset{\approx}{\overset{i_2}{\longleftarrow}} \operatorname{Im} i_3 \overset{\approx}{\longleftarrow} \cdots .$$

An inverse sequence of groups

$$G_1 \overset{i_1}{\longleftarrow} G_2 \overset{i_2}{\longleftarrow} \cdots$$

is said to be *stable* if it has a constant subsequence.

Let X be a continuum in S^n. If $S^n - X$ is connected, then $S^n - X$ has a unique end ϵ. According to Siebenmann [17, Chapter III], the fundamental group $\pi_1(\epsilon)$ *is stable* if there is a nested sequence $\{V_j\}$ of connected neighborhoods of X such that the inverse sequence

$$\pi_1(V_1 - X) \overset{i_{1*}}{\longleftarrow} \pi_1(V_2 - X) \overset{i_{2*}}{\longleftarrow} \cdots$$

is constant, where $i_q: (V_{q+1} - X) \hookrightarrow (V_q - X)$.

In this case, $\pi_1(\epsilon)$ is said to be isomorphic to Im i_{1*}.

By a *closed manifold,* we mean a compact manifold without boundary.

For definitions of regular neighborhood, PL-embedding, PL-homeomorphism, etc., we refer to Hudson [12].

A complex K in S^n (or E^n) is said to be in *trivial range* if $2 \dim K + 2 \leqslant n$.

A continuum is a compact, connected space.

Let $\{\gamma_1, \ldots, \gamma_q\}$ be a family of pairwise disjoint simple closed curves in the interior of a 2-simplex Δ^2. Let F be the closure of the component of $\Delta^2 - \bigcup_{i=1}^q \gamma_i$ which contains $\partial\Delta^2$. The components of ∂F other than $\partial\Delta^2$ are the *outermost elements* of $\{\gamma_1, \ldots, \gamma_q\}$.

3. Statement of main results.

THEOREM 1. *Let X be a globally* 1-alg *continuum in S^n, $n \geqslant 5$, having the shape of a finite complex K^k $(2k + 2 \leqslant n)$ such that $\pi_1(K)$ is abelian. If $\pi_1(K) = 0$ or $\pi_2(K) = 0$, then X has a neighborhood N, which is a regular neighborhood of a copy K_1 of K in S^n, such that $N - X \approx \partial N \times [0, 1)$ $(\approx N - K_1)$.*

As an immediate consequence of Theorem 1, Lemma 4.3 of Geoghegan and Summerhill [10], and the unknottedness of trivial-range complexes [11], we obtain the following result.

THEOREM 2. *Let X, Y be globally* 1-alg *continua in S^n, $n \geqslant 5$, having the shape of finite complexes K, L (respectively) in trivial range such that $\pi_1(K)$, $\pi_1(L)$ are abelian. If either $\pi_1(K) = \pi_1(L) = 0$ or $\pi_2(K) = \pi_2(L) = 0$, then* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *if and only if $S^n - X \approx S^n - Y$.*

COROLLARY 1. *Let X, Y be globally* 1-alg *continua in S^n, $n \geqslant 5$, having the shape of the projective space P^k, $2k + 2 \leqslant n$, $k \neq 2$. Then, $S^n - X \approx S^n - Y$.*

COROLLARY 2. *Let X be a globally* 1-alg *continuum in S^n, $n \geqslant 5$, then* $\mathrm{Sh}(X) = \mathrm{Sh}(T^k)$ *if and only if $S^n - X \approx S^n - T^k$ and X has the shape of a trivial-range finite complex K, with $\pi_2(K) = 0$ and $\pi_1(K)$ is abelian, where $2k + 2 \leqslant n$.*

REMARK 1. This corollary generalizes the weakly flat 1-spheres theorem of Daverman [8, Theorem 1].

REMARK 2. Using J. Stallings' theorem in M. A. Kervaire [14, Theorem V] and imitating an example in [16], we can construct a globally 1-alg connected polyhedron X in S^n such that $\mathrm{Sh}(X) \neq \mathrm{Sh}(S^1)$ but $S^n - S^1 \approx S^n - X$ and $\pi_2(X) \neq 0$ (even though $\pi_1(X) \approx \pi_1(S^1) \approx Z$).

THEOREM 3. *Let X be a globally* 1-alg *continuum in S^n $(n \geqslant 6)$, having the shape of a simply connected, finite complex K,* $\dim K \leqslant n - 3$. *Then, $S^n - X$ has a collar at the end ϵ, i. e. there is a PL-manifold neighborhood W of X in S^n such that $W - X \approx \partial W \times [0, 1)$.*

(Furthermore, W and K have the same homotopy type.)

COROLLARY 3. *Let X be a continuum in S^n $(n \geqslant 6)$ as in Theorem 3 above. If either $2\dim k + 1 \leqslant n$ or K is a closed PL-manifold, then $S^n - X \approx S^n - K_1$, where K_1 is the image of a PL-embedding of K into S^n.*

The following corollary follows from Irwin's embedding theorem [13] and Zeeman's unknotting theorem [20].

COROLLARY 4. *Let X and Y be globally* 1-alg *continua in S^n $(n \geqslant 6)$, which have the shape of a simply connected codimension $\geqslant 3$, closed, $(2m - n + 1)$- connected PL-manifold M^m. Then, $S^n - X \approx S^n - Y$.*

REMARK 3. For $n \geqslant 6$, Corollary 4 generalizes the weakly flat k-spheres theorem of Duvall [9, Theorem 2.1], and it also generalizes Rushing [16, Theorem 3] for $2 \leqslant k \leqslant n - 3$ and $n \geqslant 6$.

4. **Details of the proof.** Let X be a continuum in S^n. Suppose that X has the shape of a finite complex K^k. By Mardešić and Segal's definition of shape [15] (as observed [5]), we can find a cofinal sequence $\{V_i\}_{i=1}^{\infty}$ of connected neighborhoods of X in S^n with $V_{i+1} \subset V_i$, for each i, and maps $f_i: K \longrightarrow V_i$, $g_i: V_i \longrightarrow K$ such that if β^{ij} denotes the inclusion map of V_i into V_j for $i \geqslant j$, then

(1) f_i is a PL embedding for each i, if $2k + 1 \leqslant n$,
(2) $f_j g_i \simeq \beta^{ij}$ if $i > j$,
(3) $g_j \beta^{ij} f_i \simeq 1_K$, and
(4) $\beta^{ij} f_i \simeq f_j$.

LEMMA 1. *The following sequence is constant:*

$$\cdots \longrightarrow H_1(V_{j+1}) \xrightarrow{(\beta^{j+1,j})_\#} H_1(V_j) \xrightarrow{(\beta^{j,j-1})_\#} H_1(V_{j-1}) \longrightarrow \cdots$$

and $\varprojlim H_1(V_j) \approx H_1(K)$.

PROOF. The following commutative diagram

$$\begin{array}{ccccc} H_1(V_{j+1}) & \xrightarrow{(\beta^{j+1,j})_\#} & H_1(V_j) & \xrightarrow{(\beta^{j,j-1})_\#} & H_1(V_{j-1}) \\ \nearrow (f_{j+1})_\# & (g_{j+1})_\# \searrow & \nearrow (f_j)_\# & (g_j)_\# \searrow & \nearrow (f_{j-1})_\# \\ H_1(K) & \xrightarrow[1_\#]{} & H_1(K) & \xrightarrow[1_\#]{} & H_1(K) \end{array}$$

gives

(1) $(g_{j+1})_\#, (g_j)_\#$ are onto,
(2) $(f_j)_\#, (f_{j-1})_\#$ are 1-1.

We have

$$\begin{aligned}\operatorname{Im}(\beta^{j+1,j})_\# &= \operatorname{Im}[(f_j)_\# \circ (g_{j+1})_\#] = \operatorname{Im}(f_j)_\# \\ &\approx (g_j)_\#(\operatorname{Im}(f_j)_\#) = \operatorname{Im}(g_j)_\# (= H_1(K)) \\ &\qquad\qquad\qquad\qquad (\text{since } (g_j)_\#(f_j)_\# = 1_\#) \\ &\approx (f_{j-1})_\#(\operatorname{Im}(g_j)_\#) \quad \text{since } (f_{j-1})_\# \text{ is 1-1} \\ &= \operatorname{Im}(\beta^{j,j-1})_\#.\end{aligned}$$

Therefore, $(\beta^{j,j-1})_\# \mid \operatorname{Im}(\beta^{j+1,j})_\# : \operatorname{Im}(\beta^{j+1,j})_\# \longrightarrow \operatorname{Im}(\beta^{j-1,j})_\#$ is an isomorphism. That means the given sequence is constant. It is also clear that $\varprojlim H_1(V_j) \approx H_1(K)$.

LEMMA 2. *Let X be a continuum in S^n such that* $\operatorname{Sh}(X) = \operatorname{Sh}(K)$, *where K is a finite complex. Then $H_q(V, V - X) = 0$ for all $q \leqslant n - \dim K - 1$, and neighborhoods V of X in S^n.*

PROOF. Indeed, we have

$$\begin{aligned} H_q(V, V-X) &\approx \bar{H}^{n-q}(X) \quad [19, \text{Theorem } 6.2.17] \\ &\approx \check{H}^{n-q}(X) \quad [19, \text{Corollary } 6.8.8] \\ &\approx \check{H}^{n-q}(K) \quad [15, \text{Theorem } 16] \\ &= 0 \quad \text{if } n-q > \dim K \ \ (\text{i.e., } q \leqslant n - \dim K - 1).\end{aligned}$$

LEMMA 3. *Let X be a continuum in S^n having the shape of a finite complex K,* $\dim K \leqslant n-3$, *then the following sequence is constant.*

$$\cdots \longrightarrow H_1(V_{j+1} - X) \xrightarrow{(i_j)_\#} H_1(V_j - X) \longrightarrow \cdots$$

and $H_1(\epsilon) \equiv \varprojlim H_1(V_j - X) \approx H_1(K)$.

PROOF. By Lemma 2, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & H_1(V_{j+1} - X) & \xrightarrow{\approx} & H_1(V_{j+1}) & \longrightarrow 0 \\ & \Big\downarrow (i_j)_\# & & \Big\downarrow (\beta^{j+1,j})_\# & \\ 0 \longrightarrow & H_1(V_j - X) & \xrightarrow{\approx} & H_1(V_j) & \longrightarrow 0 \end{array}$$

The lemma follows easily from Lemma 1.

LEMMA 4. *Let X be a continuum in S^n having the shape of a finite complex K, with* $\dim K \leqslant n - 3$. *If X is globally* 1-alg *in S^n, then the end ϵ of $S^n - X$ is stable and $\pi_1(\epsilon) \approx H_1(K)$.*

PROOF. (The proof of this lemma is similar to the last part of the proof of Lemma 1 in [7].)

We can choose a subsequence $\{V_{j_p}\}$ of $\{V_j\}$ such that every loop in $(V_{j_{p+1}} - X)$ which is null-homologous in $(V_{j_{p+1}} - X)$ is null-homotopic in

$(V_{j_p} - X)$. Thus, we may assume the sequence $\{V_j\}$ has this property.

Using the following commutative diagram

$$\begin{array}{ccc}
\pi_1(V_{j+1} - X) & \xrightarrow{\varphi_{j+1}} & H_1(V_{j+1} - X) \\
\downarrow (i_j)_* & & \downarrow (i_j)_\# \\
\pi_1(V_j - X) & \xrightarrow{\varphi_j} & H_1(V_j - X) \\
\downarrow (i_{j-1})_* & & \downarrow (i_{j-1})_\# \\
\pi_1(V_{j-1} - X) & \xrightarrow{\varphi_{j-1}} & H_1(V_{j-1} - X)
\end{array}$$

where φ_j's are Hurewicz's homomorphisms, the globally 1-alg property implies that

$$\varphi_j \mid \mathrm{Im}(i_j)_* : \mathrm{Im}(i_j)_* \longrightarrow \mathrm{Im}(i_j)_\#$$

is an isomorphism, for each j, by the diagram chasing argument.

Therefore,

$$(i_j)_* \mid \mathrm{Im}(i_j)_* : \mathrm{Im}(i_j)_* \longrightarrow \mathrm{Im}(i_{j-1})_*$$

is an isomorphism and $\mathrm{Im}(i_j)_* \approx H_1(K)$. □

Thus, the open PL-manifold $S^n - X$ has a stable isolated end ϵ with $\pi_1(\epsilon) \approx H_1(K)$ being finitely presented. By Siebenmann [17, Theorem 3.10], there exist arbitrarily small 1-neighborhoods of ϵ, if $n \geqslant 5$; i.e., for every compact subset C of $S^n - X$, there is a neighborhood V of X in S^n such that

(1) $V \cap C = \varnothing$,

(2) the natural map $\pi_1(\epsilon) \longrightarrow \pi_1(V - X)$ is an isomorphism,

(3) the inclusion map $\partial V \subset V - X$ gives an isomorphism

$$\pi_1(\partial V) \longrightarrow \pi_1(V - X),$$

(4) ∂V and $V - X$ are connected.

LEMMA 5. *Let X be a continuum in S^n $(n \geqslant 5)$ having the shape of a finite complex K, with $\pi_1(K)$ abelian and* $\dim K \leqslant n - 3$. *If X is globally* 1-alg *in S^n, then given any neighborhood U of X in S^n, there is a neighborhood V of $X (V \subset U)$ such that*

(1) *$V - X$ is a* 1*-neighborhood of the end, and*

(2) *$i_* : \pi_1(V - X) \longrightarrow \pi_1(V)$ is an isomorphism.*

PROOF. Let W be a neighborhood of X such that

(1) $W \subset U$, and

(2) $W - X$ is a 1-neighborhood of the end.

Let $i > j$ be two integers and V a neighborhood of X such that

(1) $V \subset V_i \subset V_j \subset W \subset U$, and

(2) $V - X$ is a 1-neighborhood of the end.

(V_i, V_j satisfies conditions (2), (3), (4) preceding Lemma 1.)

First, by Lemmas 2 and 4, and the following commutative diagram, we have $i_* : \pi_1(V - X) \longrightarrow \pi_1(V)$ is 1-1, since $V - X$ is a 1-neighborhood of the end.

$$\begin{array}{ccc} \pi_1(V - X) & \xrightarrow{i_*} & \pi_1(V) \\ \approx\big\downarrow & & \big\downarrow \\ H_1(V - X) & \xrightarrow{\approx} & H_1(V) \end{array}$$

Then, $\beta_*^{VW} i_* : \pi_1(V - X) \longrightarrow \pi_1(W)$ ($\beta^{VW} : V \hookrightarrow W$) is also 1-1, by the following commutative diagram.

$$\begin{array}{ccc} \pi_1(V - X) & \xrightarrow[i_*]{} & \pi_1(V) \\ \approx\big\downarrow & & \big\downarrow \beta_*^{VW} \\ \pi_1(W - X) & \xrightarrow{1\text{-}1} & \pi_1(W) \end{array}$$

Hence $(f_j)_* (g_i \mid V)_* i_*$ is one-to-one, since $\beta_*^{VW} = (f_j)_* (g_i \mid V)_*$ by the following commutative diagram.

$$\begin{array}{ccccccc} & & & \beta_*^{VW} & & & \\ \pi_1(V) & \longrightarrow & \pi_1(V_i) & \xrightarrow{\beta_*^{ij}} & \pi_1(V_j) & \longrightarrow & \pi_1(W) \\ & {\scriptstyle (g_i \mid V)_*} \searrow & {\scriptstyle (g_i)_*} \searrow & & \nearrow {\scriptstyle (f_j)_*} & \nearrow {\scriptstyle (f_j)_*} & \\ & & & \pi_1(K) & & & \end{array}$$

We claim that $(g_i \mid V)_*$ is one-to-one. Hence, it will follow that $\pi_1(V)$ is abelian, since $\pi_1(K)$ is abelian. Thus, i_* is an isomorphism by the following commutative diagram.

$$\begin{array}{ccc} \pi_1(V - X) & \xrightarrow{i_*} & \pi_1(V) \\ \big\downarrow\approx & & \big\downarrow\approx \\ H_1(V - X) & \xrightarrow{\approx} & H_1(V) \end{array}$$

We now prove the claim. Let $\varphi : \partial\Delta^2 \longrightarrow V$ be a loop representing an element of $\ker(g_i \mid V)_*$. Then $[\varphi] \in \ker \beta_*^{VW}$ since $\beta_*^{VW} = (f_j)_* (g_i \mid V)_*$; i.e., $\varphi \simeq 0$ in W.

Let $\bar{\varphi} : \Delta^2 \longrightarrow W$ be an extension of φ over Δ^2 into W. We may assume that $\bar{\varphi}(\partial\Delta^2) \cap \partial V = \varnothing$ and $\bar{\varphi}^{-1}(\partial V)$ is a family of disjoint simple closed curves in Int Δ^2.

Let $\Gamma_1, \ldots, \Gamma_s$ be outermost loops in this family, then each $\bar{\varphi}(\Gamma_j)$ bounds a disk in W; hence $\bar{\varphi}(\Gamma_j) \simeq 0$ in W, $j = 1, \ldots, s$. Thus, $\bar{\varphi}(\Gamma_j) \simeq 0$ in $V - X$, for $j = 1, \ldots, s$, since $\bar{\varphi}(\Gamma_j) \subset V - X$ and $\beta_*^{VW} i_* : \pi_1(V - X) \longrightarrow \pi_1(W)$ is one-to-one.

Therefore, by changing the value of $\bar{\varphi}$ inside Γ_j, for $j = 1, \ldots, s$, we can define $\bar{\varphi}$ to obtain an extension of φ in V over Δ^2. In other words, $\varphi \simeq 0$ in V.

LEMMA 6. *Let X be a continuum in S^n, $n \geqslant 5$, such that*

(1) $\mathrm{Sh}(X) = \mathrm{Sh}(K)$, *where K is a finite complex with $\pi_1(K)$ abelian and* $\dim K \leqslant n - 3$,

(2) *X is globally* 1-alg *in S^n*,

(3) *Either $\pi_1(K) = 0$ or $\pi_2(K) = 0$.*

Then, given a neighborhood U of X, there is a neighborhood V of $X (V \subset U)$ such that $\pi_i(V, V - X) = 0$, for $i = 0, 1, 2$.

PROOF. Let V be a neighborhood of X as in Lemma 5.

(i) 1-connectedness is trivial, since

$$H_0(V, V - X) \approx \bar{H}^n(X) = H^n(K) = 0, \quad \text{and}$$
$$\pi_1(V - X) \xrightarrow{\approx} \pi_1(V) \longrightarrow \pi_1(V, V - X) \longrightarrow 0.$$

(ii) To show $\pi_2(V, V - X) = 0$.

Case 1. $\pi_1(K) = 0$. We have

$$\pi_1(V) \approx \pi_1(V - X) \approx \pi_1(\epsilon) \approx H_1(K) = 0.$$

Hence, the relative Hurewicz isomorphism theorem [19, Theorem 7.5.4] gives

$$\pi_2(V, V - X) \approx H_2(V, V - X) = 0.$$

Case 2. $\pi_2(K) = 0$. We choose V' to be a small neighborhood of X such that

(1) $V' \subset \mathrm{Int}\, V$,

(2) $V' - X$ is a 1-neighborhood of the end satisfying Lemma 5, and

(3) $i_* : \pi_2(V') \longrightarrow \pi_2(V)$ is trivial. (From the fact that $\pi_2(K) = 0$ and $\mathrm{Sh}(X) = \mathrm{Sh}(K)$.)

Now, let $\varphi : (\Delta^2, \partial\Delta^2) \longrightarrow (V, V - X)$ be a map representing an element of $\pi_2(V, V - X)$. We may assume $\varphi(\partial\Delta^2) \subset \partial V$.

Let T be a fine subdivision of Δ^2 so that for every $\sigma \in T$ we have $\varphi(\sigma) \subset \mathrm{Int}\, V'$, if $\varphi(\sigma) \cap X \neq \varnothing$.

We are through if we have a map $G : \Delta^2 \times [0, 1] \longrightarrow V$ satisfying

(1) $G \mid \Delta^2 \times 0 = \varphi$,

(2) $G(\Delta^2 \times 1) \subset V - X$,

(3) $G(\partial\Delta^2 \times 1) \subset V - X$.

First, we define $G : (T \times 0) \cup (T^{(1)} \times [0, 1]) \longrightarrow V$ as follows ($T^{(1)}$ is the 1-skeleton of T).

(i) $G \mid T \times 0 = \varphi$.

(ii) Let v be a vertex of T. Then

(a) $G(v \times [0, 1]) = \varphi(v)$, if $\varphi(v) \notin X$.

(b) $G(v \times [0, 1])$ is an arc α_v in V' joining $\varphi(v)$ and $G(v \times 1) \in V' - X$, if $\varphi(v) \in X$.

(iii) Let $\langle vw \rangle$ be a 1-simplex of T. Then

(a) $G(x, t) = \varphi(x)$ for every $x \in \langle vw \rangle$, $t \in [0, 1]$, if $\varphi(\langle vw \rangle) \cap X = \varnothing$.

(b) If $\varphi(\langle vw \rangle) \cap X \neq \varnothing$, then $G(\langle vw \rangle \times 1)$ will be an arc α_{vw} in $V' - X$ joining $G(v \times 1)$ and $G(w \times 1)$ such that the loop $G(\langle vw \rangle \times 0) \cup \alpha_v \cup \alpha_{vw} \cup \alpha_w$ is null-homotopic in V' (first we join $G(v \times 1)$ and $G(w \times 1)$ by an arc in $V' - X$, then we use the fact that $\pi_1(V' - X) \xrightarrow{\approx} \pi_1(V')$). Therefore, we can extend G over $\langle vw \rangle \times [0, 1]$ into V'.

Similarly for all 1-simplexes of T.

Secondly, we define $G: T \times [0, 1] \longrightarrow V$ as follows. Let σ be a 2-simplex of T.

(a) If $\varphi(\sigma) \cap X = \varnothing$, $G(x, t) = \varphi(x)$, for every $x \in \sigma$ and $t \in [0, 1]$.

(b) If $\varphi(\sigma) \cap X \neq \varnothing$, then G has been already defined on $(\sigma \times 0) \cup (\partial\sigma \times [0, 1])$ into V' with $G(\partial\sigma \times 1) \subset V' - X$. It is clear that $G | \partial\sigma \times 1 \simeq 0$ in V', hence $G | \partial\sigma \times 1 \simeq 0$ in $V' - X$. Therefore, we can extend G over $\sigma \times 1$ into $V' - X$. Thus, $G(\partial(\sigma \times [0, 1])) \subset V'$. Now G can be extended over $\sigma \times [0, 1]$ into V by the choice of V'.

Similarly for all 2-simplexes of T, we can define a map $G: T \times [0, 1] \longrightarrow V$ such that $G | T \times 0 = \varphi$ and $G(T \times 1) \subset V - X$ as we desired.

PROOF OF THEOREM 1. The proof follows by combining Lemma 6 with the following result, the proof of which is intrinsic in [5].

LEMMA 7 (CORAM, DAVERMAN, DUVALL). *Let X be a compactum in S^n, $n \geqslant 5$, which has the shape of a finite complex in the trivial range. Suppose that given a neighborhood U of X in S^n, there exists a neighborhood V such that $X \subset V \subset U$ and $\pi_i(V, V - X) = 0$ for $i = 0, 1, 2$. Then, X has a neighborhood N, which is a regular neighborhood of a copy K_1 of K in S^n, such that $N - X \approx \partial N \times [0, 1)$ $(\approx N - K_1)$.*

PROOF OF THEOREM 2. Since $S^n - X \approx S^n - K_1$ and $S^n - Y \approx S^n - L_1$ by Theorem 1, the conclusion of Theorem 2 is equivalent to saying that $K \simeq L$ if and only if $S^n - K \approx S^n - L$. The "only if" part is a special case of Theorem 1.

On the other hand, since K unknots in S^n, we may assume $K \cap L = \varnothing$. Again, since $K \cup L$ unknots in S^n, we may assume $K \cup L$ lies in S^{n-1} (the standard $(n - 1)$-sphere). The "if" part of the theorem now follows from Lemma 4.3 of [10].

REMARK 4. Combining Theorem 2.4 in [5] and the proof of Theorem 2 above, we can state the following result.

Let X, Y be continua in S^n, $n \geqslant 5$, having the shape of finite complexes in the trivial range, satisfying SLC (definition in [5]) and $\max(\dim X, \dim Y) \leqslant n-3$. Then $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ if and only if $S^n - X \approx S^n - Y$.

PROOF OF THEOREM 3. We have $\check{H}^*(X) \approx \check{H}^*(K)$, since $\mathrm{Sh}(X) = \mathrm{Sh}(K)$ [15, Theorem 16]. Hence, we can prove that $H_*(S^n - X)$ is finitely generated by using Alexander duality and the fact that $H^*(K)$ is finitely generated.

Furthermore, the end ϵ is stable and $\pi_1(\epsilon) \approx H_1(K) = 0$ (Lemma 4). We have $S^n - X$ is PL-homeomorphic to the interior of a compact PL n-manifold M [17, Theorem 5.9]. Then, we may assume that M is contained in $S^n - X$.

Let $W = S^n - \mathrm{Int}\, M$. It is clear that $W - X$ is PL-homeomorphic to $\partial W \times [0, 1)$.

Now, it is easy to see that X has a nested sequence $\{W_j\}$ of PL n-manifold neighborhoods such that $W_1 = W$ and the inclusion $W_{j+1} \hookrightarrow W_j$ is a homotopy equivalence for every $j \in N$. By terminology of [15], we can say that X is associated with the ANR-sequence $\mathbf{X} = \{W_j, i_{j+1,j}, N\}$. Then, by Theorem 6 and Theorem 5 of [15], $\mathrm{Sh}(X) = \mathrm{Sh}(W)$. Hence $W \simeq K$ by Theorem 4 in [15] and 8.6 of [1].

PROOF OF COROLLARY 3. Let $f: K \longrightarrow \mathrm{Int}\, W$ (W in the previous theorem) by a map that defines the homotopy equivalence between K and $\mathrm{Int}\, W$. We may assume that f is a PL-embedding by the following observation. If $2 \dim K + 1 \leqslant n$, f is homotopic to a PL-embedding in $\mathrm{Int}\, W$ by PL-approximation and general position theorem. If K is a closed PL-manifold and $\dim K \leqslant n - 3$, f is homotopic to a PL-embedding in $\mathrm{Int}\, W$ by Corollary 11.3.4 [21]. Therefore, in either case, f is homotopic to a PL-embedding in $\mathrm{Int}\, W$, say f'. It is clear that f' is also a homotopy equivalence. That proves the claim.

Let $K_1 = f(K)$. Then $K_1 \hookrightarrow \mathrm{Int}\, W$ is also a homotopy equivalence.

We can now apply Theorem 2.1 of [18] to conclude that $\mathrm{Int}\, W$ is PL-homeomorphic to the interior of a regular neighborhood N of K_1 in S^n, fixing K_1 ($\pi_1(\partial W) = 0 = \pi_1(\mathrm{Int}\, W)$).

Now, it can be shown that the PL n-manifold $\overline{W - N_1}$ is a H-cobordism whose boundary is $\partial W \cup \partial N_1$ with $\pi_1(\partial W) = 0$ and $\pi_1(\partial N_1) = 0$ where N_1 is a regular neighborhood of K_1 in $\mathrm{Int}\, W$. From H-cobordism theorem, we infer that $W - K_1 \approx \partial W \times [0, 1) \approx W - X$, and the corollary follows.

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